# THE PROBLEM OF THE MINIMUM IN THE QUESTION OF STABILITY OF MOTION OF A SOLID BODY WITH A LIQUID-FILLED CAVITY

(ZADACHA MINIMUMA V VOPROSE OB USTOICHIVOSTI DVIZHENIIA TVERDOGO TELA S POLOST'IU, ZAPOLNENNOI ZHIDKOST'IU)

*PMM Vol.27, No.1, 1963, pp. 16-26*

G.K. POZHARITSKII and V.V. RUMIANTSEV
(Moscow)

*(Received October 17, 1962)*

In a development of the ideas of Liapunov [1] on the stability of the equilibrium shape of a rotating liquid, it was shown in [2] that for certain conditions the question of stability of steady motion of a solid body with a liquid-filled cavity reduces to the investigation of the conditions for a minimum of the expression

$$W = \frac{1}{2}\frac{k_0^2}{S} + V$$

where $V$ is the potential energy, $S$ is the moment of inertia of the system with respect to a certain fixed axis, and $k_0$ is a constant.

For the case of equilibrium, in which $k_0 = 0$, the question of stability reduces to the problem of the minimum potential energy of the system [3]. This problem was solved in [4]; the method of solving the problem of minimum $V$ used in that paper is, with certain alterations, suitable for solution of the problem of minimum $W$ as well.

The solution of the problem of minimum $W$ is given below for a rigid body with a simply-connected cavity partially filled with liquid, in an external force field. Two examples are considered.

**1.** We imagine an absolutely rigid body having a simply-connected cavity partially filled with a homogeneous incompressible liquid. Let us assume that stationary constraints are imposed on the body, allowing it to rotate about a certain fixed straight line, which we take to be the $\zeta$-axis of a fixed rectangular system of coordinate axes $O\xi\eta\zeta$. Let the position of the rigid body relative to the coordinate system $O\xi\eta\zeta$ be defined by Lagrangean coordinates $q_1, \ldots, q_n$ $(n \leqslant 6)$. We assume that the coordinate $q_n$ defines the angle of rotation of the body about the $\zeta$-axis and is cyclic in the sense that the potential and kinetic energies of the system are independent of $q_n$; the potential energy of an element of liquid $d\tau$ has the form $\rho U_2(\xi, \eta, \zeta)d\tau$, where $\rho$ is the density.

Let us assume that in the steady-state motion of the rigid body and the liquid the coordinates are $q_i = 0$ $(i = 1, \ldots, n-1)$. We will consider in the neighborhood of this steady motion the region of variables $q_i$

$$|q_i| \leqslant H \qquad (i = 1, \ldots, n-1) \tag{1.1}$$

where $H > 0$ is a sufficiently small constant. Let $q_i$ be the coordinates of a certain fixed point belonging to the region (1.1). We will find what the form of the free surface of the liquid must be in order that for given $q_i$ the expression $W$ will have an extremum.

For the solution of this problem we take the first variation of $W$ for fixed $q_i$ and equate it to zero

$$\delta W = -\rho \int_\tau \left[\frac{1}{2}\frac{k_0^2}{S^2}\delta(\xi^2 + \eta^2) + \delta U_2(\xi, \eta, \zeta)\right] d\tau = 0$$

Here $\tau$ denotes the volume of the liquid, $U_2(\xi, \eta, \zeta)$ is the potential of the body forces acting on the liquid, $S$ is the moment of inertia of the system with respect to the axis $\zeta$ for given $q_i$ and the sought shape of the free surface of the liquid. Variation under the integral sign gives

$$\int_\tau \left[\frac{k_0^2}{S^2}(\xi\delta\xi + \eta\delta\eta) + \frac{\partial U_2}{\partial \xi}\delta\xi + \frac{\partial U_2}{\partial \eta}\delta\eta + \frac{\partial U_2}{\partial \zeta}\delta\zeta\right] d\tau = 0 \tag{1.2}$$

The variations of the coordinates of the liquid particles are related within the region $\tau$ by the continuity equation

$$\frac{\partial \delta\xi}{\partial \xi} + \frac{\partial \delta\eta}{\partial \eta} + \frac{\partial \delta\zeta}{\partial \zeta} = 0 \tag{1.3}$$

and on the wetted walls $\sigma_1$ of the cavity by the condition of no penetration

$$l\delta\xi + m\delta\eta + n\delta\zeta = 0 \tag{1.4}$$

where $l, m, n$ are the direction cosines of the outward-directed normal to $\sigma_1$. Multiplying equation (1.3) by the undetermined multiplier

$\lambda(\xi, \eta, \zeta)$ of Lagrange, integrating over the entire volume $\tau$ of the liquid and adding it to the equation (1.2), we obtain

$$\int_\tau \Big[\frac{k_0^2}{S^2}(\xi\delta\xi + \eta\delta\eta) + \frac{\partial U_2}{\partial \xi}\delta\xi + \frac{\partial U_2}{\partial \eta}\delta\eta + \frac{\partial U_2}{\partial \zeta}\delta\zeta + \\ + \lambda\Big(\frac{\partial\delta\xi}{\partial\xi} + \frac{\partial\delta\eta}{\partial\eta} + \frac{\partial\delta\zeta}{\partial\zeta}\Big)\Big] d\tau = 0$$

Since

$$\int_\tau \lambda \frac{\partial\delta\xi}{\partial\xi} d\tau = \int_\sigma \lambda l \delta\xi \, d\sigma - \int_\tau \frac{\partial\lambda}{\partial\xi}\delta\xi \, d\tau$$

then the equation written above takes the form

$$\int_\tau \Big\{\Big(\frac{k_0^2}{S^2}\xi + \frac{\partial U_2}{\partial\xi} - \frac{\partial\lambda}{\partial\xi}\Big)\delta\xi + \Big(\frac{k_0^2}{S^2}\eta + \frac{\partial U_2}{\partial\eta} - \frac{\partial\lambda}{\partial\eta}\Big)\delta\eta + \\ + \Big(\frac{\partial U_2}{\partial\zeta} - \frac{\partial\lambda}{\partial\zeta}\Big)\delta\zeta\Big\} d\tau + \int_\sigma \lambda(l\delta\xi + m\delta\eta + n\delta\zeta)\, d\sigma = 0 \qquad (1.5)$$

where $\sigma$ denotes the boundary of the region $\tau$, consisting of the wetted walls of the cavity $\sigma_1$ and the free surface of the liquid $\sigma_c$.

It is known [5] that the undetermined multiplier $\lambda(\xi, \eta, \zeta)$ may be interpreted here as the hydrodynamic pressure $p(\xi, \eta, \zeta)$. Since the pressure on the free surface of the liquid remains constant, equal to the pressure $p_0$ in the air space, while on the surface $\sigma_1$ the condition (1.4) is satisfied, then

$$\int_\sigma \lambda(l\delta\xi + m\delta\eta + n\delta\zeta)\, d\sigma = p_0 \int_{\sigma_c} (l\delta\xi + m\delta\eta + n\delta\zeta)\, d\sigma$$

In view of the incompressibility of the liquid the last integral is equal to zero. Then the equality (1.5) is possible if and only if the following equations are satisfied:

$$\frac{k_0^2}{S^2}\xi + \frac{\partial U_2}{\partial\xi} = \frac{\partial\lambda}{\partial\xi}, \qquad \frac{k_0^2}{S^2}\eta + \frac{\partial U_2}{\partial\eta} = \frac{\partial\lambda}{\partial\eta}, \qquad \frac{\partial U_2}{\partial\zeta} = \frac{\partial\lambda}{\partial\zeta} \qquad (1.6)$$

Hence we find the equation of the free surface of the liquid

$$F(\xi, \eta, \zeta) = \frac{k_0^2}{2S^2}(\xi^2 + \eta^2) + U_2(\xi, \eta, \zeta) = c \qquad (1.7)$$

giving an extremum of the expression $W$ for fixed $q_i$. The constant $c$ is determined by the quantity of liquid in a given cavity of the body.

For steady-state motion, in which $q_i = 0$ ($i = 1, \ldots, n-1$), equation (1.7) becomes the equation for the free surface of the liquid in this motion [2]

$$F_0(\xi, \eta, \zeta) = \tfrac{1}{2}\omega^2(\xi^2 + \eta^2) + U_2(\xi, \eta, \zeta) = c_0 \tag{1.8}$$

Here $\omega$ is the magnitude of the angular velocity of uniform rotation of the entire system as a single rigid body, and $k_0 = S_0\omega$, where $S_0$ is the moment of inertia of the system in the steady-state motion.

Since the liquid does not exhibit tensile resistance, the forces acting on its particles located at the free surface are directed into the liquid, and hence in steady motion the liquid must be situated on that side of the surface (1.8) where the function $F_0(\xi, \eta, \zeta) > c_0$. We denote the simply-connected region occupied by the liquid in this motion by $D_0$; for all of its points $F_0 \geqslant c_0$.

From the possible positions of the fluid for $q_i \neq 0$ we choose only those where $F(\xi, \eta, \zeta) > c$ everywhere within the liquid. This region, bounded by the surface (1.7) and the walls $\sigma_1$ of the cavity, we denote by $D$.

We will now examine the character of the extremum of the expression $W$ for fixed values of $q_i$ from the region (1.1), when the fluid occupies the region $D$. In this we will assume that the quantity $c_0$ is not the extremal of all of the values assumed by the function $F_0$ in the neighborhood of the surface $F_0 = c_0$. The locus of the point of intersection of the surface (1.8) with the walls $\sigma_1$ of the cavity describes a certain closed curve $M$, which is the boundary of the free surface of the liquid (1.8).

We imagine a unit vector $n_1(m)$ normal to the surface (1.8) at the point $m$ of the curve $M$ and directed towards the side $F_0 < c_0$, and a normal $n_2(m)$ to the surface of the walls of the cavity, directed into the cavity. We will assume that the angle $\theta(m)$ formed by these vectors varies continuously between constant limits $0 < \theta_1 \leqslant \theta(m) \leqslant \theta_2 < \pi$ during passage of the point $m$ along the curve $M$.

We also consider the following two-parameter family of surfaces

$$F = \frac{k_0^2}{2(S_0 + \Delta S)^2}(\xi^2 + \eta^2) + U_2(\xi, \eta, \zeta) = c_0 + \Delta c \tag{1.9}$$

continuous in $\Delta S$ and $\Delta c$, which are assumed to be sufficiently small in absolute value. We assume that the unit vector $n_1$ normal to the surfaces (1.9) depends continuously upon $\xi$, $\eta$, $\zeta$, $\Delta S$, $\Delta c$ in a sufficiently small neighborhood of the curve $M$, and that the vector $n_2$ depends continuously upon the point of the surface of the cavity wall in this same neighborhood.

Under these assumptions it is not difficult to prove the validity of the following assertion [4]:

For any sufficiently small $q_i$, $\Delta c$, $\Delta S$, there exists a simply-connected region $D'$, bounded by the surface of the walls of the cavity, displaced into the position $q_i$, and the surface (1.9); this region does not contain the points $F_0 < c_0 + \Delta c$ and transforms continuously into the region $D_0$ for

$$q_1^2 + q_2^2 + \ldots + q_{n-1}^2 + (\Delta c)^2 + (\Delta S)^2 \to 0$$

On the basis of what has been said it is clear that under these assumptions the constant $H$ defining the region (1.1) may always be chosen so small that for any $q_i$ from the region (1.1) the surface (1.7) belongs to the family (1.9); in this $\Delta S$ and $\Delta c$, determined by the condition of conservation of volume of the liquid, will be continuous functions of the coordinates $q_i (i = 1, \ldots, n - 1)$, vanishing for $q_i = 0 (i = 1, \ldots, n - 1)$, while the distance $l$ of surface (1.7) from surface (1.8) will not exceed $H$ within the cavity.

For certain fixed $q_i$ in region (1.1) we choose a region $D$, bounded by the walls of the cavity in the displaced position and the surface (1.7), where the constant $c = c_0 + \Delta c$, is defined. Let $\gamma < c_0 + \Delta c$ be a certain constant such that the region $D'(\gamma)$ bounded by the surface of the walls of the cavity and the surface $F = \gamma$, which transforms continuously into the region $D$ for $\gamma \to c_0 + \Delta c$, does not contain the points $F < \gamma$. We consider any possible position of the liquid completely filling the region $D'' \subset D'(\gamma)$ and find the change of the expression $W$ for fixed $q_i$ as the liquid passes from the region $D$ into the region $D''$. We have

$$W'' - W = -\rho \int_{D''} F d\tau + \rho \int_{D} F d\tau + \frac{k_0^2}{2S_0^3} (\Delta S)^2$$

This difference is positive if the region $D''$ differs from the region $D$. Actually, the difference of the first two terms represents the change of potential energy of the liquid in the force field

$$F = \frac{k_0^2}{2S^2} (\xi^2 + \eta^2) + U_2$$

for the passage of a certain portion of liquid from the region $F \geqslant c_0 + \Delta c$ into the region where $F < c_0 + \Delta c$; in this the potential energy of the liquid clearly increases. The third term is positive. Thus the location of the liquid in the region $D$ corresponds to the minimum change in the potential energy of the liquid relative to all of its possible positions in the region $D'(\gamma)$. Consequently, the lemma that follows is true.

*Lemma.* For fixed values of $q_i (i = 1, \ldots, n - 1)$ the expression $W$ has a minimum if the free surface of the liquid is defined by equation (1.7).

As a consequence of this lemma we note that for the case of equilibrium of a solid body containing a liquid, where $k_0 = 0$, the expression $W = V$ for fixed $q_i$ has a minimum [4] if the free surface of the liquid is represented by the equation

$$U_2(\xi, \eta, \zeta) = \text{const}$$

For any given set of values $q_i$ from the region (1.1), the solid body and the liquid in its cavity may be put into correspondence with a certain solid body, which we will call the transform, consisting of the given solid body and the solidified liquid with the free surface (1.7).

Then, according to the lemma, the expression $W$ for the transformed body has a minimum compared to all possible free surfaces of the liquid which are sufficiently close to (1.7).

*Theorem.* In order that the expression $W$ have a minimum in the steady-state motion of the solid body with liquid in its cavity, it is necessary and sufficient that $W$ have a minimum for $q_i = 0$ for the transformed solid body in the region (1.1).

*Proof.* For all possible transformed solid bodies in the region (1.1), let the expression $W$ have a minimum $W_0$ for $q_i = 0$ $(i = 1, \ldots, n - 1)$; then $W - W_0 > 0$. For a solid body with liquid, according to the lemma, the difference $W - W_0$ will be a still greater positive number, by which sufficiency is proved. We now prove necessity. We suppose that the expression $W$ for a rigid body with liquid has a minimum for $q_i = 0$ $(i = 1, \ldots, n - 1)$. This means [2] that for all possible sets of values of the coordinates $q_i$ $(i = 1, \ldots, n - 1)$, distances $l$ and displacements $\Delta$ such that $|q_i| \leqslant H$, $l \leqslant H$, $\Delta \geqslant \varepsilon l$, all the values assumed by the difference $W - W_0$ will remain positive and will vanish only for $q_i = 0$ $(i = 1, \ldots, n - 1)$, $l = 0$, $\Delta = 0$. Consequently, this difference will also be positive for values of the distance $l < H$, characterizing the transition from the form of the liquid in the undisturbed motion to the form determined by equation (1.7) for arbitrary $q_i$ from the region (1.1). But this also means that the expression $W$ has a minimum $W_0$ for the transformed body for $q_i = 0$. The theorem is proved.

In this manner, the problem of the minimum of the expression $W$ is reduced to the problem of the minimum of a function of a finite number of variables, which is the expression $W$ for a solid body with liquid bounded by the walls $\sigma_1$ of the cavity and the free surface (1.7).

**2.** We will find the change in the quantity $W$ for the transformed solid body in passing from the position corresponding to the steady-state motion of the system for $q_i = 0$ to a perturbed position in the region (1.1). This transition may be thought of as being carried out in two stages: (1) a displacement into the perturbed position of the whole system as a single solid body, and (2) a deformation of the shape of the liquid through the imposition on its free surface of a layer $\tau_1$, the volume of which equals zero, into a shape with the free surface (1.7).

In this the increment in the value of $W$ is represented in the form [1]

$$\Delta W = \Delta_1 W + \Delta_2 W \tag{2.1}$$

Here $\Delta_1$ is the increment for the displacement of the whole system as a rigid body into the perturbed position, while $\Delta_2$ is the increment for the subsequent deformation of the surface of the liquid into the surface (1.7). Similarly

$$\Delta S = \Delta_1 S + \Delta_2 S$$

With an accuracy to the second order in $q_i$ we have

$$\begin{gathered}\Delta_1 W = \frac{1}{2} \sum_{i,j=1}^{n-1} \left(\frac{\partial^2 W}{\partial q_i \partial q_j}\right)_0 q_i q_j + \ldots \\ \Delta_2 W = -\rho \int_{\tau_1} \left[\frac{1}{2} \omega^2 (\xi^2 + \eta^2) + U_2(\xi, \eta, \zeta)\right] d\tau + \frac{\omega^2}{2S_0} [(\Delta_2 S)^2 + 2\Delta_1 S \Delta_2 S] + \ldots\end{gathered} \tag{2.2}$$

where the index 0 denotes that the corresponding quantity is calculated for the undisturbed position of the system.

For the calculation of $\Delta_2 W$ it is convenient to introduce a moving system of coordinate axes $xyz$, rigidly attached to the solid body, the $z$-axis of which we let coincide with the $\zeta$-axis in the undisturbed position of the system.

We denote the integrand in the expression for $\Delta_2 W$, expressed as a function of $x$, $y$, $z$, by $\Phi(x, y, z, q_i)$. The equation of the free surface (1.8) of the solidified liquid in the variables $x$, $y$, $z$ has the form

$$\Phi(x, y, z, 0) = \frac{1}{2} \omega^2 (x^2 + y^2) + U_2(x, y, z) = c_0 \tag{2.3}$$

We assume that equation (2.3) may be solved uniquely for one of the variables $x$, $y$, $z$; for the sake of definiteness let it be the variable $z$. For this it is sufficient that the continuous derivative $(\partial \Phi / \partial z)_0$ does not vanish at a single point of the surface (2.3). This requirement is nonessential and is introduced only for simplification. We denote by $Q$ the region of the $xy$ plane bounded by the projection of the closed curve $M$ on this plane, where $M$ is the locus of the point of intersection of the surface (2.3) with the walls $\sigma_1$ of the cavity. The surface (1.7) in the moving axes takes the form

$$\Phi_1(x, y, z, q_i) = c \tag{2.4}$$

where the constant $c = c_0 + \Delta c$ is determined from the condition of equality of the volumes of the liquid in the cavity with the free surfaces (2.3) and (2.4). The latter condition is equivalent to the condition that the volume of the deforming layer $\tau_1$ be equal to zero

$$\int_{\tau_0} d\tau = 0$$

In the first approximation this equation is equivalent to the following

$$\iint_Q dx\, dy \int_{z_0}^{z_1} dz = 0$$

where $z_0$ and $z_1$ denote the corresponding values of the variable $z$ for points on the surfaces (2.3) and (2.4). Replacing the variable $z$ by the new variable [4] $\mu = \Phi(x, y, z, q_i) - c_0$, the latter equation to the same degree of approximation takes the form

$$\iint_Q \left(\frac{\partial z}{\partial \Phi}\right)_0 (\mu_1 - \mu_0)\, dx\, dy = 0 \tag{2.5}$$

where with an accuracy to the first order in $q_i$

$$\begin{aligned} \mu_0 &= \Phi(x, y, z_0, q_i) - c_0 = \sum_{i=1}^{n-1} \left(\frac{\partial \Phi}{\partial q_i}\right)_0 q_i + \cdots \\ \mu_1 &= \Phi(x, y, z_1, q_i) - c_0 = \Delta c + \frac{\omega^2}{S_0}(x^2 + y^2)\Delta S + \cdots \end{aligned} \tag{2.6}$$

since in the first approximation the functions $\Phi(x, y, z, q_i)$ and $\Phi_1(x, y, z, q_i)$ differ only by the term $(\omega^2/S_0)(x^2 + y^2)\Delta S$.

Substituting the values of $\mu_0$ and $\mu_1$ into (2.5), we obtain a linear equation relating $\Delta c$ and $\Delta_2 S$. A second similar equation is obtained by a calculation in the first approximation of the quantity

$$\Delta_2 S = \rho \int_{\tau_1} (\xi^2 + \eta^2)\, d\tau = \rho \iint_Q \left(\frac{\partial z}{\partial \Phi}\right)_0 (x^2 + y^2)(\mu_1 - \mu_0)\, dx\, dy \tag{2.7}$$

The equations (2.5) and (2.7), when (2.6) is taken into account, take the form

$$\iint\left(\frac{\partial z}{\partial\Phi}\right)_0\left[\Delta c+\frac{\omega^2}{S_0}\,(x^2+y^2)\,\Delta S-\sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial q_i}\right)_0 q_i\right]dx\,dy=0$$

$$\rho\iint\left(\frac{\partial z}{\partial\Phi}\right)_0(x^2+y^2)\left[\Delta c+\frac{\omega^2}{S_0}\,(x^2+y^2)\,\Delta S-\sum\left(\frac{\partial\Phi}{\partial q_i}\right)_0 q_i\right]dx\,dy=\Delta_2 S$$

and as may be seen without difficulty, they allow the unique determination of $\Delta c$ and $\Delta_2 S$ as linear functions of $q_i$. We note that if

$$\left(\frac{\partial S}{\partial q_i}\right)_0=0,\quad \iint_Q\mu_0\left(\frac{\partial z}{\partial\Phi}\right)_0 dxdy=\iint_Q\mu_0\left(\frac{\partial z}{\partial\Phi}\right)_0(x^2+y^2)\,dx\,dy=0\quad(i=1,\ldots,n-1)\tag{2.8}$$

then $\Delta c=0$, $\Delta_2 S=0$ in the first approximation.

We denote the integral in the expression for $\Delta_2 W$ by $J$. We have

$$J=\rho\int_{\tau_1}\Phi\,(x,\,y,\,z,\,q_i)\,d\tau=\frac{1}{2}\,\rho\iint_Q\left(\frac{\partial z}{\partial\Phi}\right)_0(\mu_1{}^2-\mu_0{}^2)\,dx\,dy\tag{2.9}$$

In this manner we obtain

$$\Delta_2 W=-\frac{1}{2}\,\rho\iint_Q\left(\frac{\partial z}{\partial\Phi}\right)_0(\mu_1{}^2-\mu_0{}^2)\,dx\,dy+\frac{\omega^2}{2S_0}\,[(\Delta_2 S)^2+2\Delta_1 S\,\Delta_2 S]+\cdots\tag{2.10}$$

According to the formula (2.1) the quantity $\Delta W$ is a quadratic form in the variables $q_1,\ \ldots,\ q_{n-1}$. The conditions for positive-definiteness of the latter are the conditions of minimum $W$ for a solid body with a liquid-filled cavity in a force field with a potential $V$.

**3.** *Example. Stability of rotation of a heavy solid body having a cavity containing a heavy liquid* [2]. We consider a heavy solid body with a single fixed point and having a partially liquid-filled cavity in a uniform force field. The $\zeta$-axis of the fixed coordinate system $O\xi\eta\zeta$ with origin at the fixed point $O$ of the body is directed vertically upwards, while the moving axes $x,\ y,\ z$ coincide with the principal axes of inertia of the body at the point $O$.

We denote the cosines of the angles formed by the $\zeta$-axis with the moving axes $x,\ y,\ z$ by $\gamma_1,\ \gamma_2,\ \gamma_3$, where obviously $\gamma_1{}^2+\gamma_2{}^2+\gamma_3{}^2=1$.

Let the unperturbed motion be a uniform rotation of the whole system as a single rigid body with angular velocity $\omega$ about the $z$-axis, coincident with the $\zeta$-axis, and in this rotation let the $z$-axis be a principal central axis of inertia of the system. The equation of the free surface of the liquid (2.3) takes the form

$$\Phi\,(x,\,y,\,z,\,0)=\frac{1}{2}\,\omega^2\,(x^2+y^2)-gz=c\tag{3.1}$$

where $g$ is the gravitational acceleration. In the unperturbed motion $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = 1$. In the perturbed position of the system its potential energy and moment of inertial relative to the $\zeta$-axis are, respectively

$$V = Mg(X\gamma_1 + Y\gamma_2 + Z\gamma_3) \tag{3.2}$$
$$S = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2 - 2D\gamma_3\gamma_2 - 2E\gamma_3\gamma_1 - 2F\gamma_1\gamma_2$$

Here $M$ is the mass, $X$, $Y$, $Z$ are the coordinates of the center of gravity, and $A$, $B$, $C$, $D$, $E$, $F$ are the moments of inertia and products of inertia of the system.

The function $\Phi(x, y, z, q_i)$ in the present case has the form

$$\Phi(x, y, z, \gamma_i) = \frac{1}{2}\omega^2 [x^2 + y^2 - x^2\gamma_1^2 - y^2\gamma_2^2 + z^2(\gamma_1^2 + \gamma_2^2) - 2xy\gamma_1\gamma_2 - \\ - 2z(x\gamma_1 + y\gamma_2)\sqrt{1 - \gamma_1^2 - \gamma_2^2}] - g(x\gamma_1 + y\gamma_2 + z\sqrt{1 - \gamma_1^2 - \gamma_2^2}) \tag{3.3}$$

We assume that the region $Q$ is a ring bounded by circles of radii $R_1$ and $R_2 (R_1 > R_2)$. For this, as may be easily seen, the equalities (2.8) hold and $\Delta c = 0$, $\Delta_2 S = 0$. Then

$$J = \frac{1}{2}(\gamma_1^2 + \gamma_2^2)a, \qquad a = \pi\rho g \int_{R_2}^{R_1} \left[\frac{\omega^2}{g^2}\left(\frac{\omega^2}{2}r^2 - c\right) + 1\right]^2 r^3\,dr$$

In this way, according to formula (2.1) we find

$$\Delta W = \frac{1}{2}\{[(C_0 - A_0)\omega^2 - MgZ_0 - a]\gamma_1^2 + \\ + [(C_0 - B_0)\omega^2 - MgZ_0 - a]\gamma_2^2\} + \cdots \tag{3.4}$$

The condition of minimum $W$ in this case reduces to the single inequality

$$(C_0 - A_0)\omega^2 - MgZ_0 - a > 0 \tag{3.5}$$

if it is assumed, without loss of generality, that $A_0 \geqslant B_0$.

If the liquid were weightless, then its free surface would be the surface of the circular cylinder

$$r^2 = x^2 + y^2 = b^2 \tag{3.6}$$

In place of (3.3) in this case we have

$$\Phi(r, \theta, z, \gamma_i) = \tfrac{1}{2}\omega^2 [r^2 - r^2(\cos^2\theta\gamma_1^2 + \sin^2\theta\gamma_2^2) + \tag{3.7}$$
$$+ z^2(\gamma_1^2 + \gamma_2^2) - 2r^2 \sin\theta\cos\theta\gamma_1\gamma_2 - 2rz(\cos\theta\gamma_1 + \sin\theta\gamma_2)\sqrt{1 - \gamma_1^2 - \gamma_2^2}]$$

and

$$\left(\frac{\partial\Phi}{\partial r}\right)_0 = \omega^2 b, \quad \mu_0 = -\omega^2 bz(\cos\theta\gamma_1 + \sin\theta\gamma_2) \tag{3.8}$$

We assume that the cylinder (3.6) intersects the surface $\sigma_1$ of the cavity in circles with centers on the $z$-axis at the points with coordinates $z = h + d$ and $z = h - d$. The condition of conservation of volume of the liquid in the first approximation takes the form

$$\int_{\tau_1} d\tau = \int_0^{2\pi} d\theta \int_{h-d}^{h+d} dz \int_{\mu_0}^{\mu_1} \left(r\frac{\partial r}{\partial\Phi}\right)_0 d\mu = \frac{1}{\omega^2}\int_0^{2\pi} d\theta \int_{h-d}^{h+d} \Big[\Delta c + \frac{\omega^2}{S_0} b^2\Delta_2 S +$$
$$+ \omega^2 bz(\cos\theta\gamma_1 + \sin\theta\gamma_2)\Big] dz = \frac{4\pi d}{\omega^2}\left(\Delta c + \frac{\omega^2}{S_0} b^2\Delta_2 S\right) = 0$$

Furthermore, in the first approximation we find

$$\Delta_2 S = \rho\int_{\tau_1} r^2 d\tau = \frac{4\pi d}{\omega^2} b^2\rho\left(\Delta c + \frac{\omega^2}{S_0} b^2\Delta_2 S\right)$$

From the two last equations it follows that in the first approximation

$$\Delta_2 S = 0, \quad \Delta c = 0$$

Finally, we find

$$J = \omega^2\pi b^2\rho d\,\frac{3h^2 + d^2}{3}(\gamma_1^2 + \gamma_2^2)$$

The condition of minimum $W$ in this case reduces to the single inequality [2].

$$\left(C_0 - A_0 - 2\pi\rho b^2 d\,\frac{3h^2 + d^2}{3}\right)\omega^2 - MgZ_0 > 0 \quad \text{for } A_0 \geqslant B_0 \tag{3.9}$$

**4.** *Example. The stability of steady rotation of a whirling water duct.* (All of the notation of the example is introduced independently of the foregoing). We imagine a heavy rigid body able to rotate about the vertical axis $Oz$, with a cavity having the shape of a right circular cylinder with radius $R$ and height $H$. Let the cavity be partially filled with a heavy incompressible liquid of density $\rho$ with volume $V = \epsilon\pi R^2 H$. At a certain point $A$ of the solid body let there be applied a restoring force $F$ proportional to the distance of the point $A$ from the $z$-axis and intersecting the $z$-axis at a right angle at the point $P$. We also assume that ideal constraints are applied which keep the base of the cylinder (cavity) horizontal and maintain constant the distance $OA = l$. Let $B$ be the center of gravity of the solid body with its cavity completely filled with liquid, $m$ the mass of this system, and $d^2$ the central radius of gyration about an axis parallel to the $z$-axis. Projecting $A$ and $B$ on the plane containing the fixed axes $x$ and $y$, we obtain the points $a$ and $b$. Let $\psi$ be the angle between the $x$-axis and the direction $ba$, $\varphi$ the angle between $ba$ and $Ob$, and let $Ob = r'$. Then, since the distance $ba$ is constant and equal to $e$, we obtain

$$(Oa)^2 = r'^2 + e^2 + 2r'e\cos\varphi, \quad (OP)^2 = l^2 - (Oa)^2 = l^2 - r'^2 - 2r'e\cos\varphi - e^2.$$

The varied potential energy of the system may be put into the form

$$W = \frac{k^2}{m(d^2 + r'^2) - J_p} + mg(OP) + \frac{\mu m (Oa)^2}{2} - \rho g \int_D z\,d\tau \qquad \left(J_p = \rho \int_D r'^2 d\tau\right)$$

$$(r^2 = x^2 + y^2)$$

where $\mu m$ characterizes the elastic force $F$, while $D$ is the region free of liquid.

For steady rotation with angular velocity $\omega$ about the $z$-axis the region $D$ is bounded by the paraboloid

$$z - \beta r^2 = -\alpha_1, \qquad \beta = \frac{\omega^2}{2g} \tag{4.1}$$

where $\omega^2$ satisfies the equation

$$\frac{\omega^2}{2} = \frac{k^2}{(m(d^2 + r_0'^2) - J_p)^2} \tag{4.2}$$

We give the system a possible displacement $\delta r = \xi$, $\delta\varphi = \eta$, for which the paraboloid (4.1) is displaced as a rigid body along the $z$-axis by an amount $\delta(OP)$. Its equation acquires the form

$$z - \beta r^2 = -\alpha_1 + \delta(OP) \tag{4.3}$$

and it bounds the region $D'$.

In this displacement the quantity $J_p(D) = J_p(D')$ does not change, while the gravity forces do work $m_1 g\delta(OP)$, where $m_1$ is the actual mass of the system. Equating to zero the first variation of the potential energy $\delta W$ for this displacement, we obtain

$$\delta W = m_1 g\delta(OP) - \omega^2 m r_0'^2 \xi + \mu m (Oa) = 0$$

or

$$\mu r_0' + \mu e\cos\varphi_0 - \nu g \frac{r_0' + e\cos\varphi_0}{\sqrt{l^2 - r_0'^2 - e^2 - 2r_0' e\cos\varphi_0}} - \omega^2 r_0' = 0$$

$$r_0'e \sin\varphi \left[-\mu + \frac{\nu g}{\sqrt{l^2 + r_0'^2 - e^2 - 2r_0'e\cos\varphi_0}}\right] = 0 \qquad \left(\nu = \frac{m_1}{m}\right)$$

The second equation allows the two solutions $\varphi_0 = 0$, $\varphi_0 = \pi$. For the first solution the point $b$ is closer than the point $a$ to the $z$-axis by the amount $e$, while for the second it is farther away by $e$. The solutions for $r_0'$, obtained by setting the term in the brackets equal to zero, are equal to $l$ and hence we discard them. The solution of the first equation, which goes to zero with $e$, we take in the form of a series $r_0 = a_1 e + a_2 e^2 + \ldots$ .

Assuming that $\varphi_0 = 0, \pi$, we have

$$\mu r_0' \pm \mu e - \nu g \frac{r_0' \pm e}{\sqrt{l^2 - r_0'^2 - e^2 \mp 2r_0'e}} - \omega^2 r_0' = 0$$

Restricting ourselves to the first term, we obtain

$$r_0' = \pm \frac{\nu g / l - \mu}{\mu - \nu g / l - \omega^2}\, e = a_1 e$$

The solution $\varphi_0 = 0$ proves to be possible, if $\nu g/l < \mu < \omega^2 + \nu g/l$, while $\varphi_0 = \pi$ is possible if $\mu < \nu g/l$, or if $\mu > \omega^2 + \nu g l$. We calculate now the second variation of the function $W$ for the displacement of the system mentioned above

$$\begin{aligned}\Delta_1 W = &\left[\frac{m^2 r_0'^2(\mu + 5\omega^2) + (m^2 d^2 - J_p)(\mu - \omega^2)}{m(d^2 + r_0'^2) - J_p} - \right.\\ &\left. - \frac{l^2 m_1 g}{(l^2 - r_0'^2 - e^2 - 2r_0'e\cos\varphi_0)^{3/2}}\right]\xi^2 - \\ &- r_0'e\cos\varphi_0\left[m\mu - \frac{m_1 g}{\sqrt{l^2 - r_0'^2 - e^2 - 2r_0'e\cos\varphi_0}}\right]\eta^2\end{aligned} \qquad (4.4)$$

The coefficient for $\eta^2$ is positive only for $\varphi_0 = \pi$. We calculate now the variation in the function $W$ for the transformation of the paraboloid (4.3) into the paraboloid

$$z - \beta' r^2 = -\alpha_1 + \delta(OP) - \Delta\alpha_1 \qquad \left(\beta' g = \frac{k^2}{[m(d^2 + (r_0' + \xi)^2) - J_p(D'')]^2}\right) \quad (4.5)$$

Here $D''$ is the region bounded by the paraboloid (4.5).

In the calculation mentioned above we deviate from the scheme developed in the article for convenience, since the comparison will be made not with the "frozen" surface $z - \beta r^2 = -\alpha_1$, but with the surface displaced downwards by $\delta(OP)$. This, however, is of no real importance since the variation of the function $W$ in the passage to the surface (4.3) was taken into account in formula (4.4), while all of the rest of the reasoning is unchanged when the surface of comparison is changed.

From the condition of conservation of volume we have

$$\alpha_1 = R^2\beta(1-\varepsilon) - \frac{H}{2}, \qquad \Delta\alpha_1 = R^2(1-\varepsilon)\,\Delta\beta \tag{4.6}$$

By definition

$$J_p = \rho \int\limits_{D''} r^2 d\tau = \frac{\rho\pi R^4 (1-\varepsilon)^2 H}{2} - \frac{\rho\pi H^3}{24\beta^2} \tag{4.7}$$

$$\Delta J_p = -\Delta_1 J = -\frac{\rho\pi H^3}{12\beta^3}\,\Delta\beta, \quad \Delta\beta = -\frac{2k^2}{(m(d^2+r_0'^2)-J_p)^2}\,[2mr_0'\xi - \Delta J_p] \tag{4.8}$$

Calculating the variation $\Delta_2 W$, we find

$$\Delta_2 W = -\frac{1}{2}\iint v^2 dx'dy'\,\frac{\partial z'}{\partial\Phi} - \frac{k^2}{J_0^3}(\Delta J_p)^2$$

$$v = \rho g\,[r^2 - R^2(1-\varepsilon)]\,\Delta\beta, \qquad x = r\cos\psi, \qquad y = r\sin\psi$$

where $x'y'z'$ are moving axes. From (4.7) and (4.8) we obtain

$$\left[1 + \frac{\rho\pi H^3}{\rho\beta^2\,[m(d^2+r_0'^2) - J_p]}\right]\Delta\beta = -\frac{4\beta m r_0'}{m(d^2+r_0'^2)\,J_p}\,\xi$$

Integrating and using (4.8), we obtain

$$\Delta_2 W = -\frac{2\rho\omega^2 H^3 m^2 r_0'^2}{[{}^3/_2\,\omega^4/g^2)\,[m(d^2+r'^2)-J_p] + \rho\pi H^3]\,[m(d^2+r_0'^2)-J_p]}\,\xi^2$$

Finally, a sufficient condition for stability of the regime

$$\varphi_0 = \pi \text{ for } \mu m > m\omega^2 + m_1 g/l$$

takes the form

$$\frac{m^2 r_0'^2(\mu+3\omega^2) + (md^2 - J_p)(\mu-\omega^2)}{m(d^2+r_0'^2) - J_p} - \frac{l^2 m_1 g}{(l^2 - r_0'^2 - e^2 + 2r_0'e)^{3/2}} -$$
$$- \frac{2\rho\omega^2 H^3 m^2 r_0'^2}{[3/2\,(\omega^4/g^2\,[m(d^2+r_0'^2)-J_p] + \rho\pi H^3]\,[m(d^2+r_0'^2)-J_p]} > 0$$

This problem was considered in [6] under the assumption $\omega$ = const.

## BIBLIOGRAPHY

1. Liapunov, A.M., Zadacha minimuma v odnom voprose ob ustoichivosti figur ravnovesiia vrashchaiushcheisia zhidkosti (The problem of the minimum in the stability of the equilibrium shape of a rotating liquid). *Collected Works*, Vol. 3. Akad. Nauk SSSR, 1959.

2. Rumiantsev, V.V. Ob ustoichivosti ustanovivshikhsia dvizhenii tverdykh tel s polostiami, napolnennymi zhidkost'iu (On the stability of steady-state motion of a solid body with liquid-filled cavities). *PMM* Vol. 26, No. 6, 1962.

3. Rumiantsev, V.V., Ob ustoichivosti ravnovesiia tverdogo tela, imeiushchego polosti, napolnennye zhidkost'iu (On the stability of a solid body having liquid-filled cavities). *Dokl. Akad. Nauk SSSR* Vol. 124, No. 2, 1959.

4. Pozharitskii, G.K., Zadacha minimuma v zadache ob ustoichivosti ravnovesiia tverdogo tela s chastichnym zhidkim napolneniem (The minimum problem of the stability of equilibrium of a solid body partially filled with a liquid). *PMM* Vol. 26, No. 4, 1962.

5. Lagrange, J., *Analytical Mechanics* (Russian translation), Vol. 2, Gostekhizdat, 1950.

6. Pozharitskii, G.K., Ob ustoichivosti statsionarnogo vrashcheniia karusel'nogo gidrokanala (On the stability of steady rotation of a whirling water duct). *PMTF* No. 4, 1962.


*Translated by* F.A.L.